were correct then we would expect the behavior of Z to be completely specified by the behavior of X and Y. We would not expect Z to be influenced by some other variable Q. (In economics we might argue that the effect of all excluded variables, such as Q, would be negligible). Mathematicians use the concepts of mappings, and functions to describe the relationships between variables. *You must have a very clear idea of the concepts and notations associated with* ***functions*** *if you are to do well in this course.*

2. FUNCTIONS. Mathematicians conventionally use letters like f, g, and h to name or designate *functions;* letters like x, y, and z to name *variables*; letters like a and b to name *parameters*; and letters like c and k to name *constants*. (A parameter is a constant whose value we may *choose*, e.g. we may decide to set fixed costs at the arbitrary level of $1m.) But you should bear in mind that we only have twenty-six letters in the English alphabet and so these conventions are often flouted.

Economists also have a preference for symbols that are mnemonic (look the word up if you don't know what it means!) so economists might use the letter C for the consumption and the letter c to stand for the marginal propensity to consume, I for investment. On the other hand, as economists like to say, we use Y to stand for GDP (check you ECON 207 principles text if you have forgotten what GDP stands for) and X for exports of goods and services and M for imports of goods and services.

It is ***very*** important that you thoroughly **master the symbols** that we use and have a clear idea about what they stand for (use those principles texts) because it is very easy to get confused if you start manipulating symbols and you have no idea what economic concepts the symbols represent.

MATHEMATICAL REVIEW 2

FUNCTIONS

1. VARIABLES. A *variable* is something that can take more than one value. Variables "live" in **sets** called their *domains*. The variable can take the value of any of the elements in its domain set. In this course we will only be concerned with (real) functions of real variables and so all of our variables will be represented by real numbers and so the values that they take will always be real numbers. So the domain sets for our economic variables will be the set of real numbers, R, or, more likely, the subset of R that contains only non-negative real numbers ($x \geq 0$), i.e. the set R^0.

We represent variables by symbols -- capital or lower-case letters. For example, we often use the symbol P to stand for the price of the commodity, but you should note that P is not the price itself but only a *symbol* representing the price, and that price is the theoretical price not necessarily the actual "price" that we may observe in some real world situation. (Note, for example, that prices are usually denominated in monetary units such as dollars and cents per unit of the commodity.) This will probably seem very abstract -- that is the whole point! Theories **are** abstractions (see ET 1) and *one of the functions of Econ 208* is to make you aware of the sort of implicit abstractions and assumptions that **you** are making when you are doing economics.

All theory is concerned with the **interrelationships between sets of variables** (see ET 1). A theory specifies that some variable Z is related to some other variables X and Y. This means that when X and Y change we expect Z to change. The theory specifies **qualitatively** how Z changes; e.g., Z *increases* whenever X *increases* but *decreases* whenever Y *increases*. If our theory

The investment function maps a level of real (in the economist's sense of constant price!) income (a number drawn from the set of non-negative real numbers, R^0) to the corresponding (unique) quantity invested (a number drawn from the set of non-negative real numbers, R^0). (Note that economists use the terms investment and capital to refer to physical investment and capital, **not** to financial investment and financial capital – check your ECON 207 text.)

4. **NOTATION.** There are two commonly used notations to represent functions:

(1) $$f: R \rightarrow R \quad \text{s. t.} \quad y = f(x)$$

which is read as: "f maps R, the set of real numbers, to R, the set of real numbers, such that (s.t. means "such that") y is equal to f of x (where f(x) is the "image" of x under the function or mapping f)". (We say that f is a *real-valued function of a real variable*.)

(2) $$y = f(x) \quad (x \in D_f)$$

which is read as: "y is equal to f of x, x in the domain of f" (where, in economics, D_f is usually equal to R^0).

So we would write our demand function as:

$$f: R^0 \rightarrow R^0$$

$$\text{s. t.} \quad Q^D = f(P),$$

3. **DEFINITION.** To a mathematician a function is simply a set of ordered pairs no two of which have the same first element in common. We will use the following ***definition of a function***: a **function**, f, is a mathematical object or structure that consists of ***two sets*** (the first, where the *independent* variable "lives", called the *domain* of the function (D_f), and the second, where the *dependent* variable "lives", called the *codomain* or *range* of the function (C_f)) and a ***rule*** which maps or associates **every** element in the domain with a **unique** element in the codomain.

This definition is quite abstract and may not be how you were taught to think about functions – although it is the definition that mathematicians use. You need to convince yourself that this definition agrees with what you were taught in your previous mathematics classes. Many of the problems that students have with the use of mathematics in economics are caused by their failure to really understand the idea of a function.

Some of the functions that you have already met in your ECON 206 and 207 courses are demand and supply functions, cost functions, consumption and investment functions.

The demand function maps a price (a number drawn from the set of non-negative real numbers, R^0) to the corresponding (unique) quantity demanded (a number drawn from the set of non-negative real numbers, R^0). The supply function maps a price (a number drawn from the set of non-negative real numbers, R^0) to the corresponding (unique) quantity supplied (a number drawn from the set of non-negative real numbers, R^0). The consumption function maps a level of real (in the economist's sense of constant price!) income (a number drawn from the set of non-negative real numbers, R^0) to the corresponding (unique) quantity consumed (a number drawn from the set of non-negative real numbers, R^0).

We will not usually bother to distinguish between a *function* (two sets, a rule, etc.) and its *graph* (a set of ordered pairs such that ... etc.).
The set of all functions, F, is infinite. There exist many subsets of F that have interesting properties and characteristic graphs, e.g., the set of linear functions, the set of quadratic functions, etc.

6. **FUNCTION MACHINES.** One of the best ways to think about functions is as input-output machines. We can then think of ourselves *inputting* some mathematical object, such as x or y^5 or Δz or $3x^{5-2x}$ or $\ln x^2$ or $(ax^3 - b\,x^2 + cx)^{-23x+17.4}$, into the machine and the machine then processing it and *outputting* the function image f(x) or $g(y^5)$ or $h(\Delta z)$ or $\exp(3x^{5-2x})$ or $\ln(x^2)$ or $j((ax^3 - b\,x^2 + cx)^{-23x+17.4})$. We can conceptualize what we are doing with the following schematic:

$$x \rightarrow \;|\; f \;|\; \rightarrow f(x) = y \quad (x \in R).$$

(| f | should be a box but I can't do one in Word.)

For example the rule (set of instructions or machine program) might say: take the value of x, multiply it by itself (to get x^2), then multiply what you obtain by a number, say, 5, so you get $5x^2$, then take x again and multiply it by a number, say 3, to get 3x, then subtract this from the previous calculation (we now have $5x^2 - 3x$), and then, finally, add a third number, say 4, to what you have $(5x^2 - 3x)$ to end up with $5x^2 - 3x + 4$.

With this as our rule (i.e. $x \rightarrow f(x) = 5x^2 - 3x + 4$) we can now look into our domain set (box) and take out a real number, such as 2 and then input it into our function machine to obtain a unique real number as our output, i.e. $f(2) = (y) = 5(2^2) - 3(2) + 4 = 5.4 - 3.2 + 4 = 20 - 6 + 4 = 18$. The output of the machine is a real number and lives in the co-domain set (box), i.e. $f(2) = y = 18 \in C_f$.

our supply function as

$$f: R^0 \rightarrow R^0$$

$$\text{s. t.} \quad Q^S = f(P),$$

our consumption function as

$$f: R^0 \rightarrow R^0$$

$$\text{s. t.} \quad C = f(Y),$$

our investment function as

$$f: R^0 \rightarrow R^0$$

$$\text{s. t.} \quad I = f(Y) = I_0,$$

where the I_0 signifies that we are assuming that while investment is a function of income it is a constant function and so investment is a parameter set at I_0 (say, I_0 = \$110b).

5. **DIAGRAMS.** Although *cloud diagrams* (see Figure 1) can provide a clear way to visualize functional relationships, the most common way to picture functions is to draw their **GRAPHS** in Cartesian coordinates – the familiar (x,y) coordinates you have been using for years. Technically a **graph** is a *set* of *ordered pairs* $G_f = \{(x,y): \ y = f(x) \ \ x \in Df\}$ where the *rule*, f, is some expression that allows us to specify the y coordinate *uniquely* once we know the x coordinate. The **first element** of the ordered pair (*conventionally* **x**) is located as a point on the **horizontal axis** (real line), and the **second element** (*conventionally* **y**) is located as a point on the **vertical** axis (real line). (See Figure 2.)

the algebraic parameters with numbers such as 3,4, and 5 (avoid 1 and 2 because these can lead to special cases) to aid your mathematical intuition. You may find that your algebraic skills are somewhat rusty and that the algebraic foliage is somewhat intimidating at first, but you need to get used to these simple algebraic systems in ECON 208 so that you do not end up struggling with them when you get to your upper division courses where algebra (and even simple calculus using algebraic coefficients) will become more and more common. So your new found or new honed skills will give you a large pay off in later courses and so they are worth a little cognitive dissonance.

In particular you need to bear in mind that requiring you to intellectually re-tool in this fashion is not simply some sadistic caprice on the part of the old Sleeperson, or even some arbitrary decision by the economics faculty, but rather is the consequence of two important facts about economic analysis: (1) economists almost never have reliable quantitative information about the functions they study; and (2) economists are interested in making **general** statements about the world and deriving results that hold in general not just for some particular set of numerical values of the coefficients – and we want to **prove** rather than merely **assert** that things are true.

This cognitive step **is** a major one and so you should not be dismayed if you have difficulties at first. However, you should keep reminding yourself that what you will be learning is to think in the way that professional economists do – which ought to be of interest to someone who is going to be an economics major!

7. **INVERSE FUNCTIONS.** Functions represent a sub-set of what mathematicians refer to as mappings: many-one or one-one mappings. Referring to Figure 3 we see that in part (a) each element in the domain of the function has a unique image but that

In this course the domains of the functions we encounter will always be either R or, more usually, R^0, and sometimes R^+. Note that $\sqrt{(-2)} \notin R$ and $-2 \notin R^0$ and that $0 \notin R^+$ and so $f(\sqrt{(-2)})$ and $f(-2)$ and $f(0)$ will only be defined if the domains include the relevant numbers, i.e. your function machines will not compute if you enter numbers from the wrong domain.

It is very important that you understand that **economists almost never have the luxury of dealing with functions with explicit numerical coefficients;** that our qualitative theory does not provide such information and at best we have econometric estimates of coefficients, and these are just *estimates* no matter how sophisticated our econometric estimation techniques may be. (See ET1.) Indeed, we seldom even have the sort of function with **explicit algebraic *coefficients*** and **explicit algebraic *forms*** that we use in ECON 208, i.e. the theory does **not** tell us the demand curve is linear (in fact, the standard theory of consumer choice says that demand functions cannot be linear) or that the total cost function is a cubic. We **assume** linearity of some simple algebraic form because it makes the math easier to do; all our qualitative theory says is that the price and the quantity demanded are inversely related. We write our demand function in linear form because it makes our lives simpler – and simplicity is a desirable property of any model (Albert Einstein said that theories should always be simple, but not over simplified) (see ET1). And if we do not have explicit reason for assuming a non-linear form for our functions then why not take advantage of our ignorance!

However this means that **you must master the art of dealing with functions written with algebraic coefficients**. (We could achieve even greater generality by just writing $Q^D = f(P)$ etc., but we will not do so in ECON 208 because that considerably raises the mathematical ante.) Remember that you can always replace

the elements in the domain and codomain are linked in pairs, and so every element in D_f has a unique element in C_f and *every* element C_f has a unique (reverse) element in D_f. Because the one-one mapping is "reversible" it is a function in both directions (it satisfies both the vertical and horizontal rules for a function if you can remember what that means).

The tem "reversed function" that would exactly describe what we are doing is not, unfortunately, used by mathematicians – they prefer the term "inverse function" which is rather confusing since we are not inverting the mapping (determining 1/f) but simply reversing it. Mathematicians use the notation, $\mathbf{f^{-1}}$, to denote the **inverse function** – which does **not**, as you might think, ask you to take the reciprocal of f, but asks you to **reverse the mapping** so that it takes elements from C_f (y = f(x)) and maps them to elements in D_f (x). The domain of f^{-1} is the codomain of f and the codomain of f^{-1} is the domain of f, D_f. For example if:

$$f: R \rightarrow R$$

$$y = f(x) = a + bx$$

then

$$bx = y - a$$

and

$$x = - a/b + 1/b\ y$$

but mathematician's rewrite this last expression (rather confusingly) as

$$f^{-1}: R \rightarrow R$$

$$y = f^{-1}(x) = - a/b + 1/b\ x$$

some of the images in the codomain are used more than once. For example, the first bullet in the codomain is the image of both the first and second elements in the domain of f, and the next to last bullet in C_f is the image of the fourth and fifth elements of the domain. (Note that it does not matter that the third and fifth elements of C_f are not the images of any elements on D_f because for a mapping to be a function we only require that **every** element of D_f must have a *unique* image, we do **not** require that all of the elements of C_f should be used as images of elements in D_f.) We call a mapping like that represented in Figure 3(a) a *many-one mapping* because there is at least one case in which an element in the codomain of f is the image of more than one element in the domain of f.

Turning to part (b) of Figure 3 we see an example of a mapping in which each and every element in the domain of f maps to a unique element in the codomain of f, and where if an element of C_f is used as an image of an element in D_f then it is used only once. You will also notice that in Figure 3(b) that there are the same numbers of elements in the D_f and C_f sets. Therefore the mapping in Figure 3(b) pairs each element in D_f with a unique element in C_f and vice versa. This type of mapping – where *every* element in C_f is the image of one of the elements of D_f and where *no* element in C_f is the image of more than one element in D_f – is called a *one-one* or a *one to one* mapping.

Now say we wish to reverse the mapping in figure 3(a). There are two problems. First, bullets three and five are not included in the mapping scheme (the rule underlying f is specified by the arrows between D_f and C_f). Second, bullets one and four in C_f have multiple images under the reversed mapping scheme – bullet one is connected to the first two crosses in D_f and bullet four in C_f maps to crosses four and five in D_f. On the other hand, we can reverse the one-one mapping illustrated in Figure 3(b) because

Figure 1

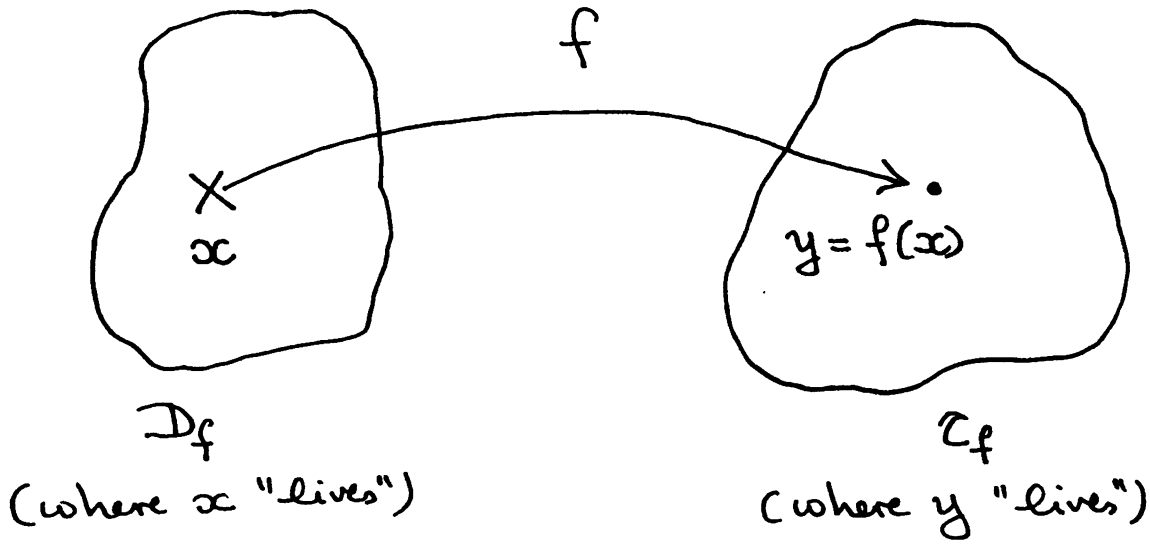
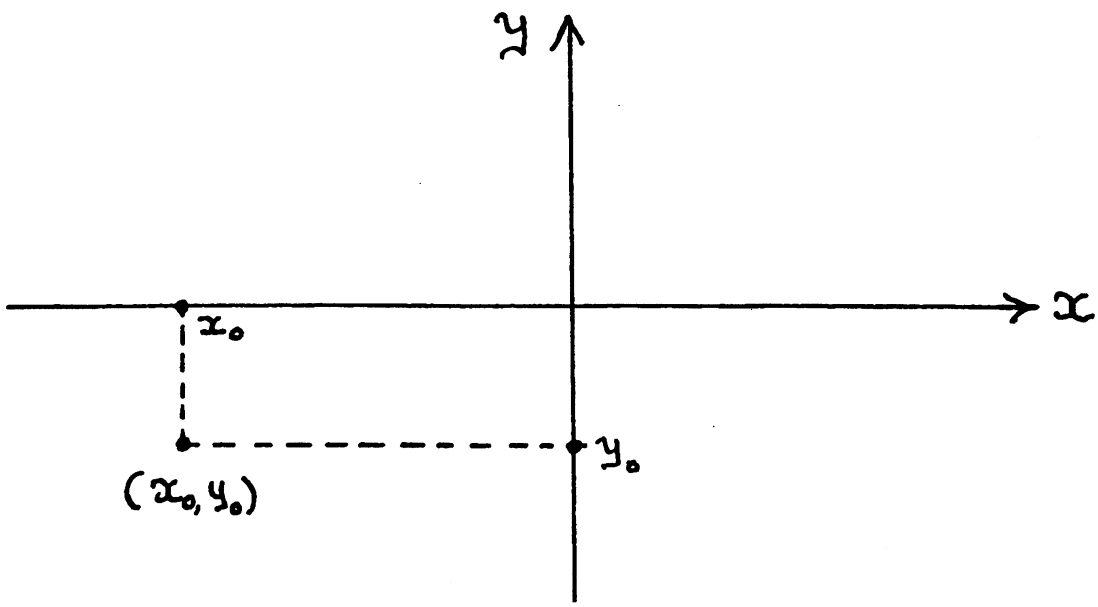


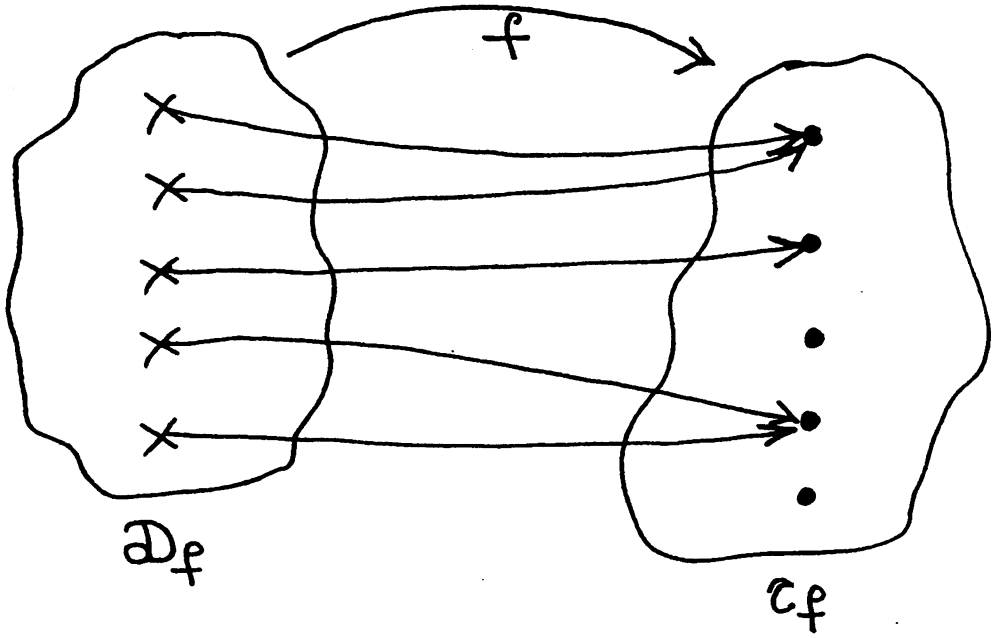
Figure 2



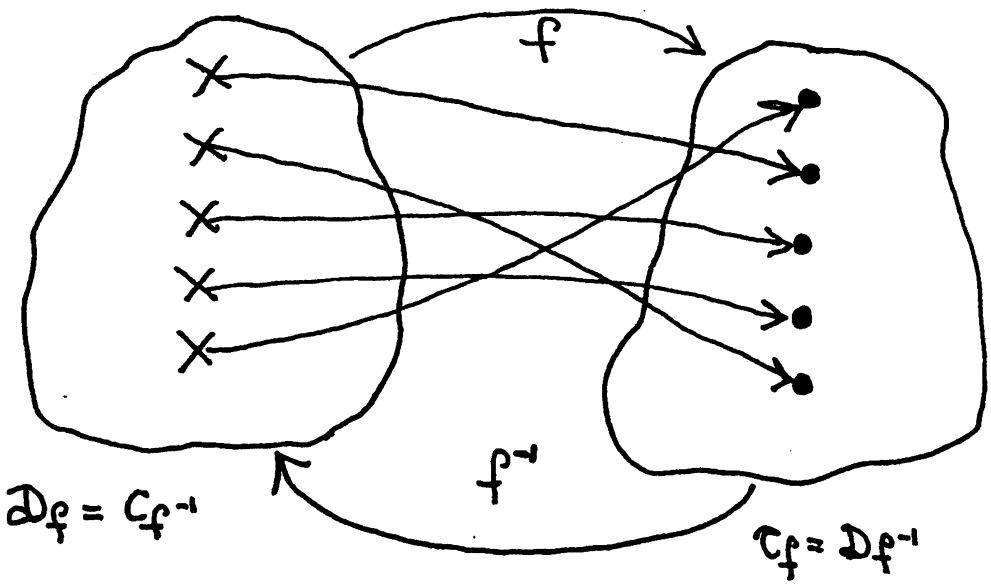
because they like to use y for the dependent variable and x for the independent variable. We will have more to say about inverse functions when we discuss linear functions in MR3.

Figure3

(a)


f
$\mathcal{D}_f$
$\mathcal{C}_f$

(b)


f
f^{-1}
$\mathcal{D}_f = \mathcal{C}_{f^{-1}}$
$\mathcal{C}_f = \mathcal{D}_{f^{-1}}$